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NUMERICAL MODELLING OF DIRECTIONAL SEAS

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ABSTRACT

A review is given of a number of numerical models of the wave elevation in directional seas. Investigations carried out in the past into the effects of wave directionality are highlighted with special emphasis on characteristics related to the numerical representation of the waves. The statistical variability of the discrete double summation model is discussed and some conclusions drawn with respect to the applicability of this commonly used representation.

1. INTRODUCTION

Detailed numerical analysis of the structural integrity of fixed structures, the motion behaviour of floating objects at sea, the loads in mooring systems and the behaviour of coastlines requires on the one hand knowledge concerning the wave dynamics and on the other hand knowledge with respect to the response of the particular system under consideration in terms of structural loads, motions, etc.

In practical cases, in order to be able to solve the total problem, simplifying assumptions are made both with respect to the nature of the wave action and the response of the system. With respect to wave loads, it has long been practice to assume that these originate from long-crested regular or irregular waves.

This assumption among others has made it possible to develop theories and computational procedures for wave loads and wave induced motions which have played a key part in the design of many of the fixed structures and floating platforms which are performing satisfactorily to this day. In reality, however, waves will always display a degree of short-crestedness which is associated with multi-directionality of the component waves.

This property of waves has been recognized for a long time but explicit inclusion of wave directionality, that is, the effect of simultaneous presence of waves from a range of directions, or even from two or more discrete directions, in the analysis of structural loads and motion behaviour has not found wide spread application as yet. This is due to a variety of reasons such as:

- Insufficient knowledge of the directional properties of the waves at the particular location of interest.
- The interaction between directional seas and the structure in terms of wave loads is not sufficiently understood.
- The complexity of the analysis is increased due to the increased number of parameters involved.
- The computational effort is considerably increased when wave directionality is included.

With regard to the computational effort required when including the effects of short-crestedness, it is to be expected that given the rate at which the computing capacity of even modest systems are developing, computing capacity as a limitation, will cease to exist. Probably more serious impediments will be the increased complexity of the analysis due to the increase in the number of parameters involved and the lack of insight in the interaction between directional seas and the structures in terms of wave loads. The lack of data regarding the wave conditions at a particular location in terms of probability of occurrence of mean wave directions, directional spreading, significant wave height, mean wave period, etc. will continue to form a final barrier to an integrated analysis of structural loads until such times that inexpensive, fast and reliable data aquisition and analysis systems are available which can monitor wave condition including wave directionality on a routine basis. This part of the total problem is the subject of another section of this symposium, however, and will not be treated further here. These aspects have been mentioned only to place the subject matter of this paper against a more general background.

The subject of this paper is numerical modelling of directional seas. The numerical model used to describe the wave field is of importance since the properties of this model affect the subsequently calculated quantities such as, for instance, wave loads and wave induced motions. It is therefore not surprising that the choice of the numerical model is affected by the nature of the phenomena being investigated. For instance, if the purpose of the investigation is to determine wave frequency oscillatory loads on a structure we may only require that the random wave elevation at the structure conforms with a given significant wave height and mean period and that furthermore the wave elevations conform with a Gaussian model. When considering the low frequency behaviour of a moored vessel under influence of slowly varying second order wave drift forces we are interested in reproducing the correct wave grouping in the incident irregular waves, since it is this property in the waves which determines to a large extent the behaviour of low frequency second order wave drift forces. Clearly, such considerations can affect the choice of the numerical model to a considerable degree.

In the following section the general form of some of the models which can be used to describe such seas will be reviewed briefly. After subsequently reviewing a number of applications of numerical models of directional seas, the statistical properties of a commonly used formulation will be discussed.

2. MATHEMATICAL MODELS

The basic assumption underlying almost all of the numerical models used to date is that the real sea surface elevation is a zero mean stationary, ergodic, random Gaussian process. The statistical properties are independent of time and the co-ordinates in the horizontal plane. The surface is considered as the result of linear superposition of an infinity of independent regular long-crested waves approaching from all directions:

$$\zeta(t,\bar{x}) = \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} \zeta(\omega_i,\psi_k,t,\bar{x}) \quad \ldots\ldots\ldots\ldots\ldots \quad (1)$$

This model is consistent with a description of the wave motion based on linear potential theory, i.e. the first order term of the power series expansion of the total potential describing the flow.

It is customary to characterize a directional irregular sea condition in terms of a directional wave spectrum, $S_\zeta(\omega,\psi)$. The directional spectrum is related to the wave components through the following relationship:

$$\lim_{\Delta\omega,\Delta\psi\to 0} S_\zeta(\omega_i,\psi_k)\,\Delta\omega\,\Delta\psi = \overline{\zeta(\omega_i,\psi_k,t,\bar{x})^2} \quad \ldots\ldots\ldots \quad (2)$$

The numerical models which can be used in analyses involving directional seas fall into two categories, i.e. the frequency domain characterization based on the directional spectrum and time domain representations.

With respect to the frequency domain directional wave spectrum, two general formulations are used:

$$a \;:\; S_\zeta(\omega,\psi) = S_\zeta(\omega).f(\psi)$$

$$b \;:\; S_\zeta(\omega,\psi) = S_\zeta(\omega).g(\omega,\psi) \quad \ldots\ldots\ldots\ldots\ldots \quad (3)$$

in which $S_\zeta(\omega)$ is the point wave spectrum and the functions $f(\psi)$ and $g(\omega,\psi)$ are so-called spreading functions.

Various formulations for $f(\psi)$ and $g(\omega,\psi)$ have been proposed and investigated on their merits. See, for instance, Mitsuyasu et al [1], Hasselman et al [2], Holthuijsen [3], Kuik and Holthuijsen [4]. The actual form chosen for $S_\zeta(\omega,\psi)$ is, however, not directly relevant for the discussion in this paper, so this subject will not be pursued further here.

Time domain representations of directional waves are, with respect to the contributions from different directons, generally of the discrete summation type. The basic formulation is as follows:

$$\zeta(t,\bar{x}) = \sum_{k=1}^{M} \zeta_k(t,\bar{x}) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

The numerical models can differ in the description of the irregular long-crested seas from the k-th direction.

In principle, the irregular long-crested wave trains approaching from the k-th direction can be generated using the same techniques as have been used in the past for irregular long-crested seas. Strictly speaking, only one of these methods has been used in investigations involving directional seas. The other methods will be mentioned here for the sake of completeness. The methods can be classed in two groups, i.e. models based on discrete summation of harmonic components and models based on digital filtering of white noise.

A realization for $\zeta_k(t,\bar{x})$ can be based on the following discrete summations, both of which are due to Rice [5]:

$$\text{(a)} \quad \zeta_k(t,\bar{x}) = \sum_{i=1}^{N} \zeta_{ik} \cos(\bar{k}_i.\bar{x} - \omega_i t + \underline{\varepsilon}_{ik})$$

$$\text{(b)} \quad \zeta_k(t,\bar{x}) = \sum_{i=1}^{N} \{a_{ik} \cos(\bar{k}_i.\bar{x} - \omega_i t) + b_{ik} \sin(\bar{k}_i.\bar{x} - \omega_i t)\} \quad \ldots\ldots\ldots\ldots\ldots (5)$$

in which:

$$\zeta_{ik} = \sqrt{2\, S_\zeta(\omega_i,\psi_k)\, \Delta\omega\, \Delta\psi}$$

$\underline{\varepsilon}_{ik}$ = uniformly distributed random phase angle $\quad \ldots\ldots\ldots\ldots\ldots (6)$

and a_{ik} and b_{ik} are drawn from a Gaussian distributed process with variance equal to:

$$\sigma^2_{ik} = S_\zeta(\omega_i,\psi_k)\, \Delta\omega\, \Delta\psi \quad \ldots\ldots\ldots\ldots\ldots (7)$$

The properties of both representations of a random irregular sea have been investigated among others by Tuah and Hudspeth [6]. The first of these is the model which is most extensively used to generate realizations of directional seas. The difference in these representations is that model (a) results in realizations for $\zeta_k(t,\bar{x})$ of which the spectral density is identical for all realizations while the spectrum obtained from model (b) is a stochastic quantity.

In both models the frequencies are usually chosen equidistant. If periodicity of the realization is a problem this is avoided by choosing non-equidistant frequencies.

Ergodic stationary random realizations of $\zeta_k(t)$ can be generated using filtered white noise. Examples of formulations are:

$$\text{(a)} \quad \zeta_k(t) = \int_{-\infty}^{\infty} w(t-\tau)\, h(\tau)\, d\tau$$

$$\text{(b)} \quad \zeta_k(t) = -\sum_{n=1}^{N} a_n\, \zeta_k(t - n.\Delta t) + \sum_{m=0}^{M} b_m\, w(t - m.\Delta t) \quad \ldots\ldots\ldots\ldots\ldots (8)$$

Equation (8) (a) represents straightforward digital filtering of a white noise input while (b) represents an auto-regressive moving average (ARMA) filter. The ARMA filter technique has been investigated among others by Samii and Vandiver [7].

From equations (8) (a) or (b) the wave elevations $\zeta_k(t,\bar{x})$ at other points in the horizontal plane can be computed using digital filters which incorporate wave dispersion effects. See for instance Burke and Tighe [8].

In this paper, the time domain representation of the wave elevation based on equation (4) and equation (5) (a) will be denoted as the double summation or double Fourier series representation.

3. APPLICATIONS

In this section some recent applications of the mentioned numerical descriptions of an irregular directional sea will be highlighted.

Battjes [9] discussed the effect of directional spreading of the incident wave field on the transverse loads on a long structure. Making use of the frequency domain concept of the directional wave spectrum $S_\zeta(\omega,\psi)$, and a linear, wave frequency and wave direction dependent transfer function for the wave loads, it was shown that when directional spreading of the incident waves is taken into account, the computed horizontal wave loads per unit length of the structure are reduced. An example of the reduction is shown in Fig. 1.

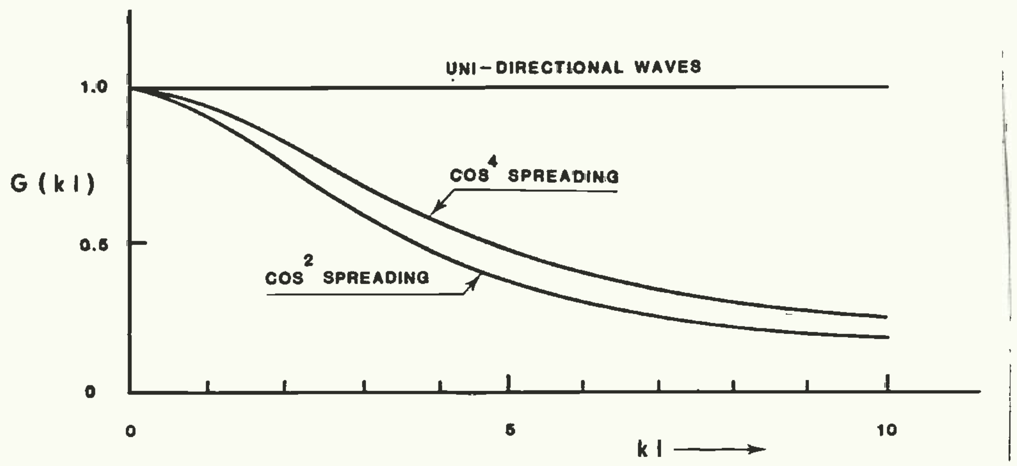

Fig. 1: Reduction factor for wave loads on a vertical wall section with length 2ℓ in beam seas. From Battjes [9].

Dallinga, Aalbers and v.d. Vegt [10] have determined the influence of directional spreading in the incident waves on the motions of an offshore transport barge and on the loads in the seafastenings holding a jacket structure on the barge. The influence of directional spreading was investigated for a range of mean wave directions relative to the barge using the frequency domain directional spectrum and linear transfer functions for the loads and motions. It was found that directional spreading increased the roll motions of the barge in head seas and, in beam seas, tended to lead to a small reduction in roll motions relative to the case of long-crested waves. See Fig. 2.

Also working in the frequency domain, Molin and Fauveau [11] have investigated the influence of directional spreading on low frequency second order horizontal forces on structures which are associated with the second order wave potential. In order to clarify the effect of wave spreading, the undisturbed low frequency horizontal water particle acceleration in the wave is determined.

This quantity is representative for inertia dominated wave loads on structures. Use is made of the following type of expression:

$$S_a(\mu) = 2 \int_0^\infty \int_0^{2\pi} \int_0^{2\pi} S_\zeta(\omega,\psi)\, S_\zeta(\omega+\mu,\alpha)\, \gamma_a^{(2)}(\omega,\psi,\omega+\mu,\alpha)\, d\psi\, d\alpha\, d\mu \qquad (9)$$

where $\gamma_a^{(2)}$ is the quadratic transfer function of the particle acceleration determined from potential theory.

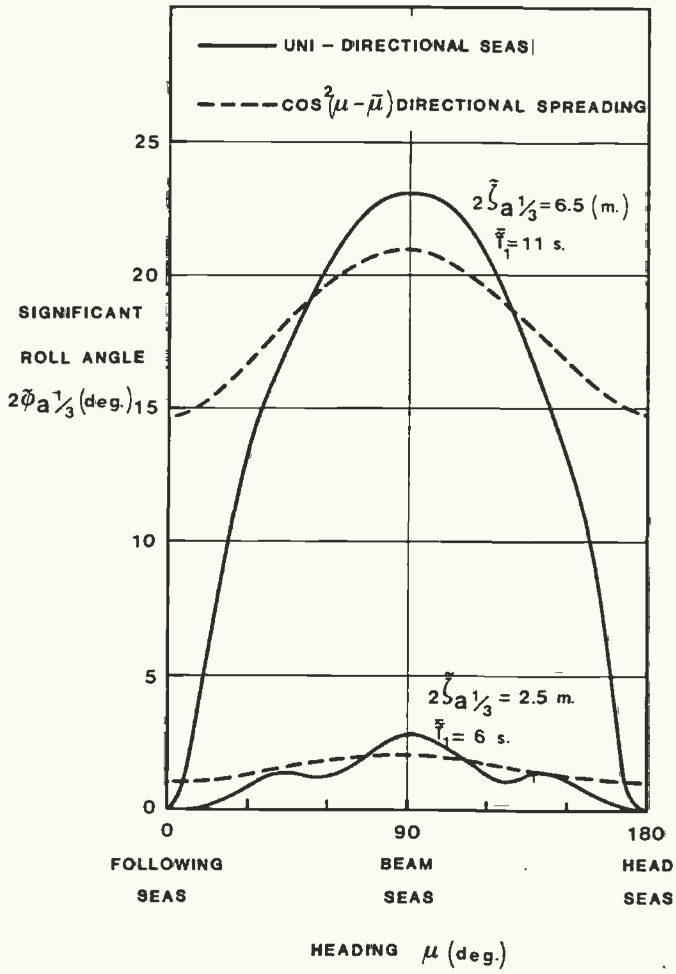

Fig. 2: Roll motions of a transport barge in beam seas. From Dallinga et al [10].

Molin and Fauveau concluded that low frequency second order inertia loads were significantly effected by directional spreading of waves. It was stressed, however, that this contribution formed only a part of the total second order wave loads.

Borgman and Yfantis [12] have derived expressions for the spectra and cross-spectra of the horizontal components of the forces on jacket structures in directional seas based on Morison's equation and the directional wave spectrum.

From the aforegoing it will be clear that a frequency domain approach does not always require that the quantity under consideration be a linear process. However, when the process considered is of a general non-linear nature, the effect of directional spreading is generally investigated based on time domain descriptions of directional waves.

Hackley [13] simulated the loads on a pile in directionally spread waves in the time domain based on the application of Morison's equation.

Wave kinematics were computed as a function of time based on a double summation representation of random directional waves which was evaluated using the inverse Fast Fourier Transformation technique. See for instance Borgman [14].

Hackley concluded that directional spreading causes a significant transfer of energy from the in-line to the normal direction of wave propagation and tends to reduce the in-line RMS and maximum values of forces. This conclusion agrees with the findings of Molin and Fauveau.

Teigen [15] has carried out simulation computations in the time domain when investigating the influence of directional spreading on the behaviour of a tension leg platform including the effect of wave frequency loads and low frequency wave drift forces. Two-dimensional linear potential theory was used to calculate first order wave load transfer functions. Maruo's formula [16] was used to compute mean drift forces and Newman's approximation [17] was applied to obtain time records of the drift forces.

The effect of directional spreading was approximated by first carrying out an angular averaging procedure for the first order transfer functions and the drift force coefficients. The resulting response was then calculated as for long-crested seas. In effect therefore, the directional spreading was incorporated in the transfer functions and not in the wave train itself, which remained long-crested. It was concluded that directional spreading of the incident waves tended to decrease the in-line horizontal motions of the platform slightly and increase the transverse motions.

Marthinsen [18] has investigated the influence of directional seas on low frequency wave drift forces on moored vessels using the double summation model. The instantaneous direction of travel of the local wave field is determined and the drift force subsequently based on drift force coefficients valid for uni-directional waves.

Time domain simulations of second order effects in the incident directional waves have been discussed by Dean and Sharma in [19] and [20]. Such effects have also been discussed by Sand [21] and Sand et al [22].

Lambrakos [23] investigated the wave loads and movements of a pipeline lying on the sea floor. The directional seas were represented by a double Fourier series. The amplitudes of the regular wave components were based on a Bretschneider frequency spectrum and a cosine squared spreading function. To describe the sea surface 51 frequencies and 21 directions were used. Lambrakos concluded that wave induced pipeline movements are a strong function of wave direction, but not a strong function of wave spreading within reasonable limits.

A notable phenomenon mentioned by Lambrakos is that double Fourier series representation for the sea surface did not yield the same significant wave height over the length of the pipe, see Fig. 3. It was concluded that this had considerable effect on the results of computations.

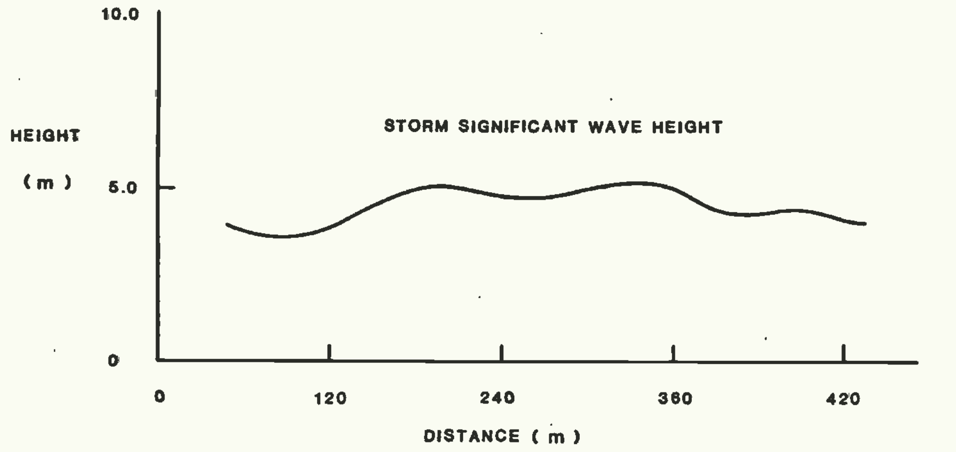

Fig. 3: Variation of significant wave height along the length of a pipeline. From Lambrakos [23].

Forristal [24] also made use of the double Fourier series representation for the wave elevation while investigating the kinematics of directionally spread waves. He noted that this representation yielded spectra of the time series of the wave elevation which were not equal to the input spectra. The output spectra are random variables because wave energy from several directions appears at each frequency in the summations. The total energy at each frequency finally depends on the phase angles of the wave components. Forristal noted that the randomness of the simulated wave spectrum is not actually an imperfection in the model since the spectrum of natural ocean waves is also random.

Goda [25] in a study regarding the directional spectral resolutions of arrays of wave probes simulated the incident waves also using the double Fourier series representation. He suggested that the frequencies of the regular wave components should be chosen to be non-correlating. See also Borgman [26]. This avoids the wave train repeating itself within the simulation time. Goda concluded that the directional wave field should be simulated using no less than 100 frequency components and no less than 36 directional components over an angular spread of 180 degrees. In a discussion on the statistical variability of the double summation representation Goda states that conclusions regarding the effects of wave directionality should be based on at least 10 simulation runs even when only the mean values of output quantities are considered. When considering statistical properties of the output, such as the standard deviation, more than 100 runs could be necessary.

4. STATISTICAL VARIABILITY OF THE NUMERICAL MODEL

From a review of the studies carried out, two aspects of the numerical representation of a directional sea in the time domain stand out namely, the statistical variability of the realizations and the computational effort required to evaluate the double summations.

Tucker [26] and Tucker, Challenor and Carter [27] investigated the statistical variability of realizations of irregular waves based on the summation of discrete components as given in equation (5) (b) for the contribution from the k-th direction. As will be shown in this section, this is also the same type of representation as is found for the total wave elevation $\zeta(t)$ based on the double summation formulation given by equation (4) and equation (5) (a). He obtained the following expression for the mean and variance of the variance of such realizations:

$$E[m] = \int_0^\infty S_\zeta(\omega)\, d\omega$$

$$\sigma_m^2 \rightarrow \frac{2\pi}{T} \int_0^\infty S_\zeta^2(\omega)\, d\omega \quad \ldots\ldots\ldots\ldots\ldots (10)$$

This indicates that the statistical error of such realizations is larger for narrow band spectra.

It was concluded that the simulated wave trains were not ergodic and did not simulate a random Gaussian process and hence do not correctly simulate ocean waves. This seems to contradict Flower [28] who concluded that a discrete summation very soon approaches a random Gaussian process as the number of discrete components increases. Furthermore, Tucker et al [27] concluded that the wave grouping is incorrectly reproduced in such simulations. See also Goda [25]. According to Tucker et al simulating waves by filtering Gaussian white noise avoids these problems at the expense of greater computing costs.

In order to clarify some of these aspects we will determine the statistical variability of the spectral density of the wave elevation and of the wave groups of realizations based on the double summation representations according to equation (4) and equation (5) (a). To illustrate these aspects it is sufficient to regard the wave elevation in one point of the horizontal plane. The elevation is given by:

$$\zeta(t) = \sum_{i=1}^{N} \sum_{k=1}^{M} \zeta_{ik} \cos(\omega_i t + \varepsilon_{ik}) \quad \ldots\ldots\ldots\ldots (11)$$

The amplitudes ζ_{ik} follow from:

$$\zeta_{ik} = \sqrt{2\, S_\zeta(\omega_i, \psi_k)\, \Delta\omega\, \Delta\psi} \quad \ldots\ldots\ldots\ldots\ldots (12)$$

Assuming that all frequencies approach from all directions, equation (11) can also be written as follows:

$$\zeta(t) = \sum_{i=1}^{N} (a_i \cos \omega_i t + b_i \sin \omega_i t) \quad \ldots\ldots\ldots (13)$$

with:

$$a_i = \sum_{k=1}^{M} \zeta_{ik} \cos \underline{\varepsilon}_{ik}$$

$$b_i = \sum_{k=1}^{M} \zeta_{ik} \sin \underline{\varepsilon}_{ik} \quad \ldots\ldots\ldots\ldots (14)$$

a_i and b_i are independent zero mean random Gaussian variables with variance equal to:

$$\sigma_a^2 = \sigma_b^2 = \sum_{k=1}^{M} \tfrac{1}{2} \zeta_{ik}^2$$

$$= \sum_{k=1}^{M} S_\zeta(\omega_i, \psi_k) \Delta\psi \, \Delta\omega$$

$$= S_\zeta(\omega_i) \Delta\omega \quad \ldots\ldots\ldots\ldots (15)$$

4.1. Spectral density of realizations

The spectral density of a discrete double summation consists of a series of delta-functions at the discrete frequencies and directions. In this paper, however, we prefer to consider the average spectral density over the frequency and direction intervals $\Delta\omega$ and $\Delta\psi$ with mid-frequency ω_i and mid-direction ψ_k respectively.

The spectral density at frequency ω_i of a particular realization of $\zeta(t)$ can be found by taking the mean square or variance of the wave elevation component with frequency ω_i as follows:

$$S(\omega_i) \Delta\omega = \overline{(a_i \cos \omega_i t + b_i \sin \omega_i t)^2}$$

$$= \sum_{k=1}^{M} \sum_{\ell=1}^{M} \tfrac{1}{2} \zeta_{ik} \zeta_{i\ell} \cos(\underline{\varepsilon}_{ik} - \underline{\varepsilon}_{i\ell}) \quad \ldots\ldots (16)$$

This shows that the spectral density of a realization depends on the random phase angles $\underline{\varepsilon}_{ik}$. The ensemble mean is found by averaging over all realizations:

$$E[S(\omega_i) \Delta\omega] = \sum_{k=1}^{M} \tfrac{1}{2} \zeta_{ik}^2 \quad \ldots\ldots\ldots\ldots (17)$$

This shows that the time average over a realization does not yield the ensemble average value found for the spectral density $S(\omega_i)$. This confirms that such realizations are not ergodic in this respect.

The variance of the spectral density $S(\omega_i)$ is found from:

$$\sigma_s^2 = E[S(\omega_i)^2] - E^2[S(\omega_i)] \quad \ldots\ldots\ldots\ldots (18)$$

Taking into account equation (16) and equation (17) this yields:

$$\sigma_s^2 = E^2[S(\omega_i)] \quad \ldots\ldots\ldots\ldots (19)$$

This result conforms with the variance of the spectral density of a random Gaussian process with a continuous spectrum. See for instance ref. [29]. This property is therefore not influenced by the discretization.

The variance of the wave elevation $\zeta(t)$ for a realization is found from the mean square of equation (11). This becomes:

$$\sigma_\zeta^2 = \sum_{i=1}^{N} \sum_{k=1}^{M} \sum_{\ell=1}^{M} \tfrac{1}{2} \zeta_{ik} \zeta_{i\ell} \cos(\underline{\varepsilon}_{ik} - \underline{\varepsilon}_{i\ell}) \qquad \ldots\ldots (20)$$

Which again shows that the result is influenced by the choice of the phase angles.

The ensemble mean of the variance is:

$$E[\sigma_\zeta^2] = \sum_{i=1}^{N} \sum_{k=1}^{M} \tfrac{1}{2} \zeta_{ik}^2$$

$$= \sum_{i=1}^{N} \sum_{k=1}^{M} S_\zeta(\omega_i, \psi_k)\, \Delta\psi\, \Delta\omega \qquad \ldots\ldots (21)$$

For large N and M:

$$E[\sigma_\zeta^2] \rightarrow \int_0^\infty \int_0^{2\pi} S_\zeta(\omega,\psi)\, d\psi\, d\omega$$

$$= \int_0^\infty S_\zeta(\omega)\, d\omega \qquad \ldots\ldots (22)$$

This corresponds with the variance of the input spectrum.

The variance of the variance given by equation (20) can be found by application of equation (18):

$$\sigma_{\sigma^2}^2 = \sum_{i=1}^{N} \sum_{k=1}^{M} \sum_{\ell=1}^{M} \tfrac{1}{4} \zeta_{ik}^2 \zeta_{i\ell}^2 \qquad \ldots\ldots (23)$$

For large N and M this becomes:

$$\sigma_{\sigma^2}^2 \rightarrow \Delta\omega \int_0^\infty \int_0^{2\pi} \int_0^{2\pi} S_\zeta(\omega,\psi)\, S_\zeta(\omega,\alpha)\, d\psi\, d\alpha\, d\omega$$

$$= \Delta\omega \int_0^\infty S_\zeta^2(\omega)\, d\omega \qquad \ldots\ldots (24)$$

This corresponds with the result given by Tucker [26]. Obviously, decreasing $\Delta\omega$, thus increasing the number of frequency components N will reduce the variance of the variance. For $N \rightarrow \infty$, $\sigma_{\sigma^2}^2 \rightarrow 0$ and this type of representation becomes ergodic.

4.2. Wave grouping: spectrum of square of wave envelope

Finally we will look at some of the properties of the wave grouping of such realizations. Specifically, we will consider the variability of spectral density of the square of the wave envelope. We could also discuss the wave envelope itself, however, the square of the wave envelope is of direct importance for the behaviour of low frequency second order wave loads on floating structures and the second order set-down effects in the incident waves. See Pinkster and Huijsmans [30].

The wave elevation given by equation (11) can also be written as follows:

$$\zeta(t) = A(t)\cos\{\omega_0 t + \varepsilon(t)\} \qquad \ldots\ldots\ldots\ldots\ldots \text{(25)}$$

in which the envelope is given by:

$$A^2(t) = \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{M}\sum_{\ell=1}^{M} \zeta_{ik}\,\zeta_{j\ell}\cos\{(\omega_i - \omega_j)t + (\underline{\varepsilon}_{ik} - \underline{\varepsilon}_{j\ell})\} \qquad \ldots\ldots\ldots\ldots\ldots \text{(26)}$$

and $\varepsilon(t)$ by:

$$\tan\varepsilon(t) = \frac{\sum_{i=1}^{N}\sum_{k=1}^{M} \zeta_{ik}\sin\{(\omega_i - \omega_0)t + \underline{\varepsilon}_{ik}\}}{\sum_{i=1}^{N}\sum_{k=1}^{M} \zeta_{ik}\cos\{(\omega_i - \omega_0)t + \underline{\varepsilon}_{ik}\}} \qquad \ldots\ldots \text{(27)}$$

The spectral density of $A^2(t)$ at frequency ω_n is found by taking the mean square of the components with frequency ω_n. The following result is found:

$$S_{A^2}(\omega_n) = \frac{2}{\Delta\omega}\sum_{j=1}^{N-n}\sum_{q=1}^{N-n}\Big[\sum_{k=1}^{M}\sum_{r=1}^{M}\sum_{\ell=1}^{M}\sum_{s=1}^{M} \zeta_{j+n,k}\,\zeta_{q+n,r}\,\zeta_{j\ell}\,\zeta_{qs}\cdot$$
$$\cdot\cos\{(\underline{\varepsilon}_{j+n,k} - \underline{\varepsilon}_{j\ell}) - (\underline{\varepsilon}_{q+n,r} - \underline{\varepsilon}_{qs})\}\Big] \qquad \ldots \text{(28)}$$

This result applies to a particular realization.

The ensemble mean is found to be:

$$E[S_{A^2}(\omega_n)] = \frac{2}{\Delta\omega}\sum_{j=1}^{N-n}\Big[\sum_{k=1}^{M}\sum_{\ell=1}^{M}\zeta^2_{j+n,k}\,\zeta^2_{j\ell}\Big] \qquad \ldots\ldots \text{(29)}$$

For large N and M and applying equation (12) we again find the result for a continuous spectrum:

$$S_{A^2}(\omega_n) \rightarrow 8\int_0^\infty\int_0^{2\pi}\int_0^{2\pi} S_\zeta(\omega+\omega_n,\alpha)\,S_\zeta(\omega,\beta)\,d\alpha\,d\beta\,d\omega$$
$$= 8\int_0^\infty S_\zeta(\omega+\omega_n)\,S_\zeta(\omega)\,d\omega \qquad \ldots\ldots\ldots\ldots \text{(30)}$$

We will use equation (30) to clarify the result found for the variance of the variance of the wave elevation given by equation (24).

The variance of the wave elevation corresponds to the mean square value, i.e. the constant part of the square of the wave elevation. Besides a constant part, the square of the wave elevation $\zeta(t)$ contains sum frequency and difference frequency components.

The difference frequency components are related to the square of the wave envelope as follows:

$$\zeta^2_{\ell.f}(t) = \tfrac{1}{2} A^2(t) \quad \ldots\ldots (31)$$

The spectrum of the low frequency part of the square of the wave elevation is therefore:

$$S_{\zeta^2_{\ell.f}}(\omega_n) = 2 \int_0^\infty S_\zeta(\omega+\omega_n)\, S_\zeta(\omega)\, d\omega \quad \ldots\ldots (32)$$

which for $\omega_n = 0$ becomes:

$$S_{\zeta^2_{\ell.f}}(0) = 2 \int_0^\infty S^2_\zeta(\omega)\, d\omega \quad \ldots\ldots (33)$$

Substitution in equation (24) gives:

$$\sigma^2_{\sigma^2} = \tfrac{1}{2}\Delta\omega\, S_{\zeta^2_{\ell.f}}(0) \quad \ldots\ldots (34)$$

This result shows that the variance of the variance of the wave elevation corresponds with the area under the spectrum of $\zeta^2_{\ell.f}(t)$ in the frequency band of 0 to $\tfrac{1}{2}\Delta\omega$. This is exactly that part of the variance of the $\zeta^2_{\ell.f}(t)$ process which is not accounted for by the oscillatory components of the discrete summation representation. This is illustrated in Fig. 4. As Tucker [27] pointed out, the variance of the variance of $\zeta(t)$ is related to the spectral width. The above discussion shows that it can also be related to the behaviour of the spectrum of the square of the wave envelope for difference frequencies tending to zero.

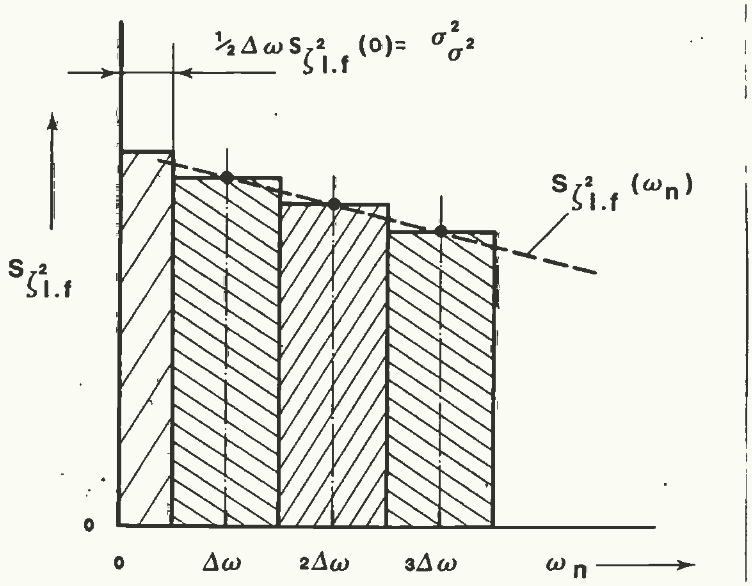

Fig. 4 Relationship between the variance of the variance of the wave elevation and the spectrum of the square of the wave elevation.

Finally we will pay attention to the variance of the spectral density of the square of the wave envelope at frequency ω_n.

Starting from equation (28) and applying equation (18) finally yields:

$$\sigma^2_{S_{A^2}} = E^2[S_{A^2}(\omega_n)] + \Delta\omega . f\{S_\zeta(\omega+\omega_n), S_\zeta(\omega), S_\zeta(\omega-\omega_n)\} \quad . . (35)$$

in which f{ } is a complicated, positive function of the point spectral density S_ζ.

For large N (smaller $\Delta\omega$):

$$\sigma^2_{S_{A^2}} \rightarrow E^2[S_{A^2}(\omega_n)] \quad . . . . (36)$$

This is the same result as found for the spectrum $S_\zeta(\omega)$. See equation (19).

For the discrete case, equation (35) shows that a double summation representation yields a higher variance for the spectrum of the envelope squared than is found for the continuous case. Decreasing the frequency step $\Delta\omega$, however, also decreases this additional effect. Equation (36), incidentally, also indicates that the statistical variability of wave groups is not larger than that of the waves themselves.

The aforegoing discussion shows that the discrete summation representation of an irregular directional sea introduces additional statistical variability with respect to the variance of the wave elevation record and the spectral density of the square of the wave envelope when compared with results valid for a random Gaussian process with a continuous spectrum. It is also seen that the additional variability can be reduced to arbitrarily small values by increasing the number of frequency components.

Equation (24) and equation (25) which give the variance of the variance of the wave elevation and the variance of the spectrum of the square of the wave envelope respectively, can be evaluated prior to carrying out simulations. This can be an aid in selecting an appropriate value for the frequency step $\Delta\omega$. The selection of an appropriate directional step $\Delta\psi$ can be based on comparison of the various discrete summations over the wave direction with the outcome of the integrations valid for the continuous cases.

In this section we have considered the statistical variability of a discrete double summation representation of directional waves. With regard to such aspects of representations based on filtered white noise or the ARMA method it can be stated that, since such models generate random signals with continuous spectra, the statistical variability will conform with the random Gaussian model.

4.3. Some effects of the simulation output on the numerical model

The aforegoing remarks on the selection of appropriate values for $\Delta\omega$ and $\Delta\psi$ for a discrete summation representation apply when the aim of the simulations is to simulate properties of random directional waves themselves. In many cases, however, the aim of the simulations is to obtain data on, for instance, wave induced loads on a fixed structure or the slow motions of a moored vessel induced by wave drift forces. In such cases we should choose the numerical model of the directional seas such that, for instance, the wave drift force records contains a sufficient number of frequencies so that low frequency resonant response of the horizontal motions of a moored vessel are modelled correctly. This effect has been studied in the frequency domain by Dacunha, Hogben and Standing [31], see Fig. 5. This may require a much smaller frequency step Δω than would be deemed necessary to model the properties of the wave field itself.

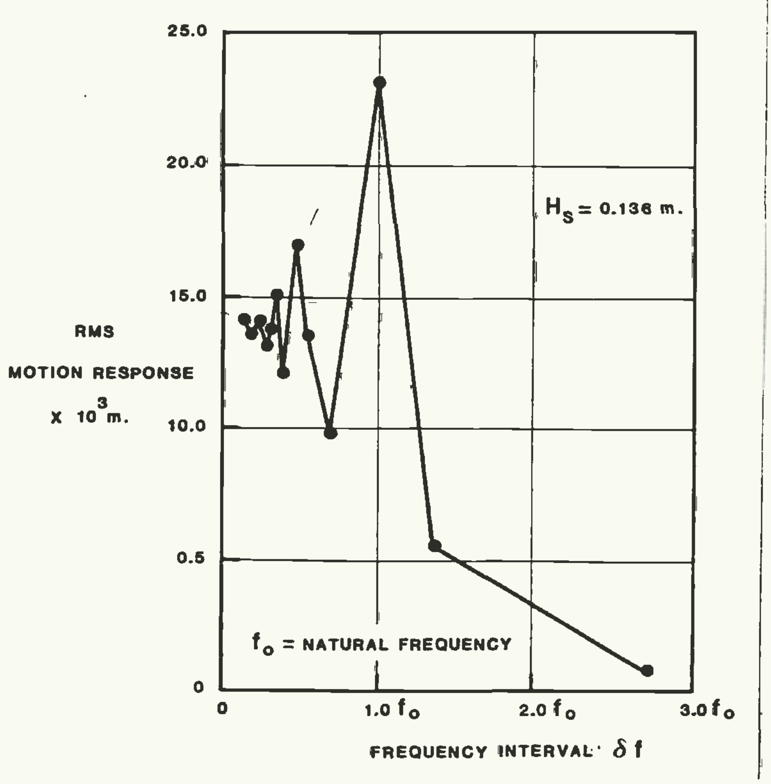

Fig. 5 Influence of the frequency interval of the wave spectra on the RMS of the low frequency horizontal motions of a vessel induced by second order wave drift forces. From Dacunha et al [31].

A factor also influencing the choice of the frequency interval of a discrete summation representation is the duration of the time domain simulation. If the frequencies are chosen equidistant with frequency interval Δω, the wave train and all quantities derived from it repeat after a time period equal to $2\pi/\Delta\omega$. Therefore, if for some reason a particular duration of the simulation is required, and we do not wish the wave train to repeat within the simulation period, this will result in additional requirements with respect to the frequency interval Δω.

With regard to the choice of the directional step Δψ it can be added that the sensitivity of a structure with respect to the wave directions may be such that a higher directional resolution is required than would appear when comparing simulated wave properties based on a discrete summation model with input data based on a continuous spectrum.

4.4. Numerical evaluation of time records

A numerical model of irregular directional waves which is based on a double summation requires a considerable amount of computational effort.

A double summation involving N discrete wave frequencies and M discrete directions requires the evaluation of M•N sine functions at each time step. See equation (11). If the same wave frequencies approach from all directions, the double summation is replaced by a single summation by first carrying out summations over the wave directions. Subsequent summations over the wave frequencies for each time step can be made efficiently through the application of Fast Fourier Transformations. Alternatively, in order to evaluate the sine functions at consecutive time steps, use can also be made of recursion relationships, see Lambrakos [23] and Goda [25]. Quantities which are linearly related to the wave elevation, such as the wave particle velocities, etc. are treated in a manner similar to the wave elevation. Quantities which are non-linearly related to the wave elevation, such as the low frequency second order wave drift forces on a moored vessel, or the second order wave set-down in the incident waves present special problems with respect to numerical evaluation in the time domain.

The discrete summation representation of such phenomena which is consistent with equation (11) for the incoming waves is of the following type:

$$F^{(2)}(t) = \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{k=1}^{M} \sum_{\ell=1}^{M} \zeta_{ik} \zeta_{j\ell} T^{(2)}_{ijk\ell} \cdot$$

$$\cdot \cos\{(\omega_i - \omega_j)t + (\underline{\varepsilon}_{ik} - \underline{\varepsilon}_{j\ell})\} \qquad \ldots\ldots (37)$$

Such expressions can be reduced to a single summation provided it is assumed that all frequencies approach from all directions and that the wave frequencies are equally spaced. See, for instance, ref. [30]. Evaluation of such single summations can again be carried out efficiently through the application of FFT or through the use of recursion relationships.

5. CONCLUSIONS

As more reliable data on the directional properties of waves at sea become available, the desire to investigate the effects of directionality on wave kinematics, wave induced loads and motions increases. As computational capacity increases, it will be possible to carry out not only frequency domain analyses but also extensive time domain analyses.

From a survey of investigations carried out in the past it appears that, in time domain analyses, the discrete double Fourier series representation is most often used to describe the wave field. Although other representations may also be used, the double Fourier series is chosen since it is a logical extension of the well known single Fourier series so often used in connection with studies involving uni-directional seas. As noticed by a number of investigators, this representation for directional waves appears to possess a degree of statistical variability which on the one hand is accepted as also being a property of real waves and on the other hand could be a basis for rejection of this type of representation.

In this paper the statistical variability of this type of realization is regarded in relation to the spectral density of the wave elevation and to the square of the wave envelope. The latter quantity contains information on wave grouping. It appears that the double Fourier series representation in some respects possesses greater statistical variability than may be expected for real seas. On the other hand, the additional variability is directly related to the frequency interval of the discrete representation. By a suitable choice of this parameter the additional variability due to the discretization can, at the expense of computational costs, be reduced to arbitrarily small values.

NOMENCLATURE

ζ	:	wave elevation
$\underline{t}$	:	time
$\underline{x}$	:	position vector in horizontal plane
$\underline{k}$	:	wave number vector
i,j,q,n	:	wave frequency indices
k,ℓ,r,s	:	wave direction indices
$\Delta\omega$	:	frequency step
$\Delta\psi$	:	wave direction step
$\underline{\varepsilon}$	:	random phase angle
σ	:	root-mean square
w(t)	:	white noise
a_n, b_m	:	ARMA coefficients
$S_\zeta(\omega,\psi)$	:	input directional spectrum
$S_\zeta(\omega)$	:	input point spectrum
α,β,ψ	:	wave directions
ω,μ	:	frequency
T	:	length of simulation record in s
$S(\omega_i,\psi_k)$	:	directional spectrum of discrete representation
$S(\omega_i)$	:	point spectrum of discrete representation
ζ_{ik}	:	amplitude of wave component with frequency ω_i from direction ψ_k
$T^{(2)}_{ijk\ell}$	:	quadratic transfer function of second order drift forces or set-down

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