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Velocity Distribution in Open Channels

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THE von Kármán universal logarithmic velocity distribution law for pipes is

$$\frac{V - V_{max}}{\sqrt{\frac{\tau_0}{\rho}}} = \frac{2.3}{k} \log_{10} \frac{y}{r_0} \quad \ldots\ldots\ldots\ldots (1)$$

where V is the velocity at a radial distance y from the wall; V_{max} is the maximum velocity in the cross section, that is, the velocity at the center; τ_0 is the friction or shear stress at the wall; ρ is the mass density of the fluid; k is a universal constant having a value of 0.4; r_0 is the radius of the pipe, and 2.3 is merely the factor for conversion from common to natural logarithms. For the case of uniform two-dimensional open-channel flow the above equation becomes

$$\frac{V - V_{max}}{\sqrt{gdS}} = \frac{2.3}{k} \log_{10} \frac{y}{d} \quad \ldots\ldots\ldots\ldots (2)$$

where d is the depth of the flow, S the slope of the channel, and g the acceleration of gravity. Equation 2 can be solved for V, the velocity at any level, which in turn can be integrated over the depth to give the discharge per unit width. This quantity divided by the depth, d, gives the average velocity,

$$\overline{V} = V_{max} + \frac{2.3}{k} \sqrt{gdS}\ \frac{1}{d} \int_{\delta}^{d} \log \frac{y}{d} dy \quad \ldots (3)$$

where δ, the lower limit of integration, is taken as a small distance from the bottom. Performing the integration in Eq. 3 and noting that the lower limit of the integral vanishes as δ approaches zero gives,

$$\overline{V} = V_{max} - \frac{1}{k} \sqrt{gdS} \quad \ldots\ldots\ldots\ldots (4)$$

Eliminating V_{max} between Eqs. 2 and 4, and rearranging, gives

$$V = \overline{V} + \frac{1}{k} \sqrt{gdS} \left(1 + 2.3 \log_{10} \frac{y}{d}\right) \quad \ldots (5)$$

which expresses the distribution law in terms of $\overline{V}$ instead of V_{max}.

The location of the point at which the velocity is equal to the average is found by substituting $\overline{V}$ of Eq. 4 for V in Eq. 2, with the result that

$$y_a = \frac{d}{e} = 0.368\, d \quad \ldots\ldots\ldots\ldots (6)$$

where y_a, in precise language, is the distance from the channel bottom to the filament moving with a velocity, $\overline{V}$, equal to the average for the profile section, and e is the base of the natural logarithms. The depth to this filament measured from the stream surface is then $d - y_a = 0.632\ d$. The above result states that the depth to the average velocity is always the same fraction of the depth of the flow, that is, 0.632 d. This result is valid as long as the velocity distribution is logarithmic in form and is not affected by the values of $\sqrt{gdS}$ or k in Eq. 2, or by other factors such as the channel roughness. For instance the result would not be changed by varying k, provided that in doing so the velocity distribution was not altered from the logarithmic form.

Figure 1 shows rectangular and semi-logarithmic plots of velocity profile measurements made on the center line of a rectangular flume 2.77 ft wide, with uniform flow 0.59 ft deep. The measured values are represented by circles while the solid lines represent Eq. 2. Since the velocity distribution follows Eq. 2, the velocity at 0.632 d from the surface is equal to the average for the profile section.

Experience in stream gaging has shown that the velocity at a depth of 0.6 d from the surface is a good approximation of the average for the profile section. From data on 476 measurements in rivers, Hoyt and Grover (*River Discharge*, Wiley and Sons, 1912) obtained a mean value of 0.62 d for the depth to the average velocity. It has also been found that the mean of the velocities at 0.2 d and 0.8 d gives a good approximation of the average. For logarithmic distribution the depth to the average velocity is 0.632 d instead of 0.6 d, while the average of the velocities at 0.2 d and 0.8 d is exactly equal to that

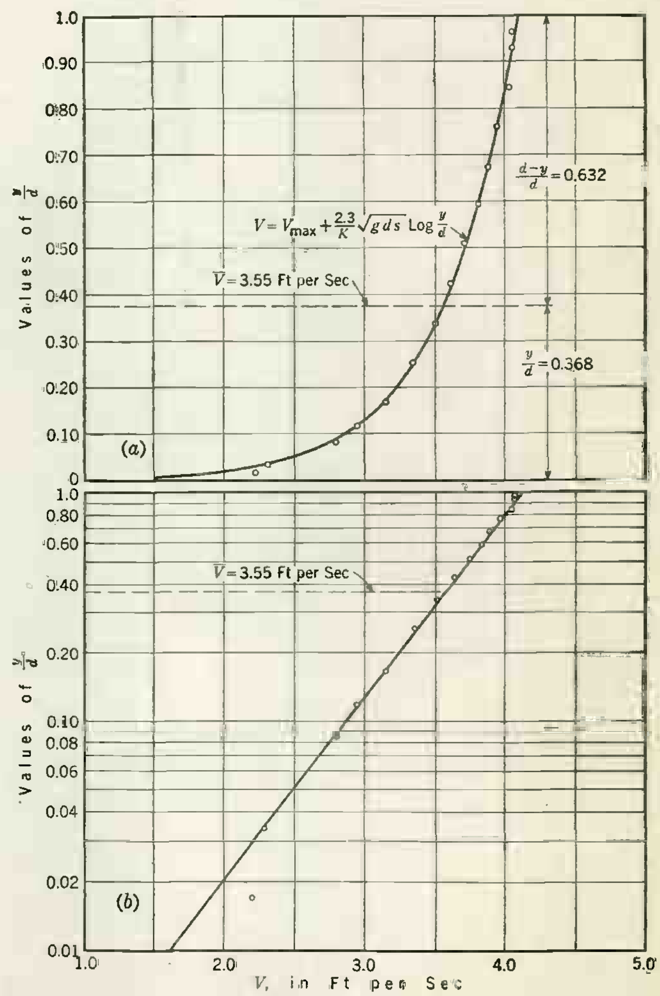

FIG. 1. VELOCITY PROFILE AT CENTER OF A FLUME 2.77 FT WIDE FOR A FLOW 0.59 FT DEEP

at depth 0.6 d. The latter relationship may be seen from Fig. 1 (b) or derived from Eq. 2.

The fact that in streams the maximum velocity does not occur at the surface precludes the possibility that the velocity distribution is strictly logarithmic. However, the relations that have been discussed indicate a striking similarity between observed distributions and those following the logarithmic law, and offer reasonable justification for the use of this law in calculating some of the performance characteristics of natural streams.

Relations similar to those developed for two-dimensional channels can also be derived for circular pipes. Bakhmeteff (*The Mechanics of Turbulent Flow*, Princeton University Press, 1936) obtained the relationship

$$\frac{V_{max} - \overline{V}}{\sqrt{\tau_0/\rho}} = \frac{3}{2}\,\frac{1}{k} \quad \ldots\ldots (7)$$

in which $\overline{V}$ is now the average velocity in the pipe.

Substituting $\overline{V}$ from Eq. 7 for V in Eq. 1 gives

$$y_a = r_0 e^{-3/2} = 0.223\, r_0 \quad \ldots\ldots (8a)$$

or

$$r_a = r_0 - y_a = 0.777\, r_0 \quad \ldots\ldots (8b)$$

where y_a is the radial distance from the pipe wall to the point where the local velocity has the same value as the average for the entire cross section, and r_a is the distance from the center to the same point. Exhaustive experiments with flow in pipes have shown that the velocity distribution follows the logarithmic law but the relationship expressed in Eq. 8 has not been pointed out.